THEORY OF THE FLOW OF AN ELASTOVISCOPLASTIC MEDIUM

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We present a rheological equation of state for an elastoviscoplastic medium which agrees well with experimental data from the flow of consistent lubricants. We obtain an equation for the stationary flow of an elastoviscoplastic medium.

Numerous experimental data for the flow of viscoplastic dispersed systems and colloidal solutions indicate that the flow of these media does not conform to the law currently applied for describing the flow of a Schwedoff–Bingham fluid. Therefore rheological equations of state have been proposed involving a nonlinear dependence between the tangential stress and the velocity gradient for viscoplastic materials. Among these equations we note those due to Herschell and Bulkley [1], Casson [2], and Shul'man [3]. In [4] a method was developed for approximating the nonlinear flow equations of consistent lubricants in the form of the linear Schwedoff–Bingham equations.

At the same time, studies have shown that nonlinear viscoplastic materials, consistent lubricants [5], structured petroleum products [6], solutions and polymer melts [7] possess elastic properties and that for such media shear moduli and Young's moduli have been determined.

Thus an urgent need exists to formulate a theory for the flow of elastoviscoplastic media which would take into account in the flow rules for viscoplastic materials the influence of elastic properties. It is therefore necessary, in the case of elastoviscoplastic media, to verify, under stationary flow conditions, a direct relationship to the Hencky [8] and Il'yushin [9] models in which a linear relationship exists between the tangential stresses and the velocities of deformation, i.e., it is necessary to show that the Schwedoff–Bingham equations are applicable.

We establish a rheological equation of state for an elastoviscoplastic medium based on the structural rheological model shown in Fig. 1. This model consists of a spring, simulating the elastic properties of the medium, in series with a combination viscoplastic element. We obtain the rheological equation of state by combining velocities of deformation, represented by elements of the model in series, with stresses, represented by elements of the model in parallel.

Equations for the deformation of an elastic element comply with Hooke's law,

$$p_{ij} = \frac{3\sigma}{1+\sigma}\, p_c\delta + 2G\varepsilon_{ij}. \tag{1}$$

The behavior of the plastic element of the model is described by the flow equations for a rigid-plastic body,

$$p_{ij} = p_c\delta + 2\,\frac{\theta}{\lambda}\,\dot{\varepsilon}_{ij}, \tag{2}$$

while the flow equations for the viscous element of the model are those for Newtonian flow

$$p_{ij} = p_c\delta + 2\eta\dot{\varepsilon}_{ij}. \tag{3}$$

Combining the stresses for the plastic and viscous elements, we obtain the flow equations for a viscoplastic element:

$$p_{ij} = p_c\delta + 2\left(\frac{\theta}{\lambda} + \eta\right)\dot{\varepsilon}_{ij}. \tag{4}$$

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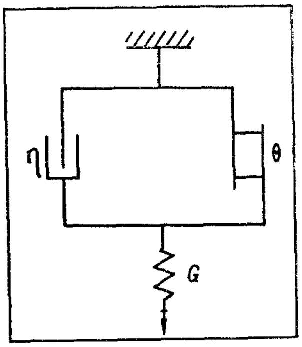

Fig. 1

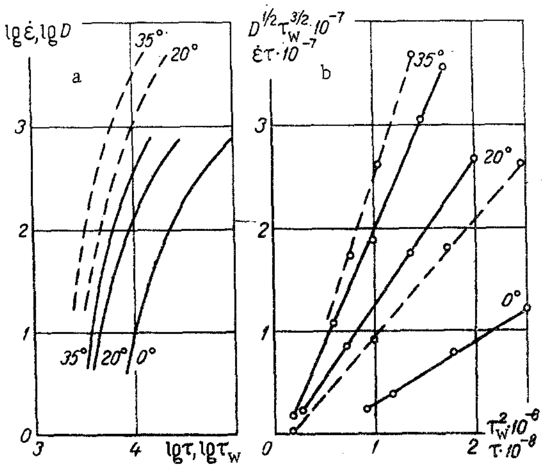

Fig. 2

Fig. 1. A model for an elastoviscoplastic medium.

Fig. 2. Flow of an aliphatic lubricant when studied in (a) a rotation viscosimeter, and (b) a capillary viscosimeter. [In part a of the figure the solid curves are those for $\log D = f(\log \tau_W)$, the dashed for $\log \dot{\varepsilon} = f(\log \tau)$; in part b of the figure the solid curves are those for $D^{1/2}\tau_W^{3/2} = f(\tau_W^2)$, the dashed for $\dot{\varepsilon}\tau = f(\tau^2)$.]

The velocities of elastic deformation may be obtained from Eq. (1):

$$\dot{\varepsilon}_{ij} = \frac{1}{2G}\dot{p}_{ij} - \frac{3\sigma}{2(1+\sigma)G}\dot{p}_c\delta. \quad (5)$$

We obtain the rheological equation of state for an elastoviscoplastic medium by adding together the corresponding velocities of deformation for the elastic [Eq. (5)] and viscoplastic [Eq. (4)] elements of the model (see Fig. 1):

$$p_{ij} - p_c\delta + \frac{1}{G}\left(\frac{\theta}{\lambda} + \eta\right)\left(\dot{p}_{ij} - \frac{3\sigma}{1+\sigma}\dot{p}_c\delta\right) = 2\left(\frac{\theta}{\lambda} + \eta\right)\dot{e}_{ij}. \quad (6)$$

We obtain the scalar factor from the Mises plasticity condition, to which the plastic element of the model is subject:

$$\frac{1}{2} s_{ij}s_{ij} = \theta^2. \quad (7)$$

Differentiating this expression with respect to the time, we find

$$s_{ij}\dot{s}_{ij} = 0. \quad (8)$$

Since the viscous element of the model has no effect on the transition from elastic to plastic deformations, we may put $\eta = 0$ in Eq. (6) to get

$$\dot{p}_{ij} = \frac{3\sigma}{1+\sigma}\dot{p}_c\delta - \lambda\frac{G}{\theta}(p_{ij} - p_c\delta) + 2G\dot{e}_{ij}.$$

Substituting these values into Eq. (8), we determine the scalar factor in the form

$$\lambda = \frac{1}{\theta} s_{ij}\dot{e}_{ij}. \quad (9)$$

Consequently, λ depends not only on the velocities of the deformations but also on the stresses, this being also a consequence of the plasticity theory due to Reuss [10].

A stationary flow of an elastoviscoplastic medium will exist if the time of flow of the medium is large, $t \to \infty$, or if the relaxation period of the medium is small, $\eta / G \to 0$. The equation of state (6) may be written in the form

$$p_{ij} = p_c\delta + 2\left(\frac{\theta}{\lambda} + \eta\right)\dot{e}_{ij}. \tag{10}$$

From relation (9) and the Eqs. (10), we obtain the scalar factor for a stationary flow:

$$\lambda = \frac{s_{ij}}{\theta}\,\frac{I_2'}{2\dot{e}_{ij}} \quad \text{(no summation)}, \tag{11}$$

from Eqs. (10) and (11) we find the equations for the stationary flow of an elastoviscoplastic medium, namely,

$$s_{ij}^2 = 2\left(2\theta^2\frac{\dot{e}_{ij}}{I_2'} + \eta s_{ij}\right)\dot{e}_{ij} \quad \text{(no summation)}. \tag{12}$$

For pure shear the equation for the stationary flow of an elastoviscoplastic medium assumes the form

$$\tau^2 = \theta^2 + \eta\tau\dot{\varepsilon}. \tag{13}$$

Thus the presence of elastic properties in a viscoplastic medium makes its stationary flow rule nonlinear; consequently, this medium does not follow the Hencky–Il'yushin postulate concerning linearity of the flow rule.

We compare the stationary flow rule (13) with experimental data relating to the flow of a consistent lubricant (oil). Figure 2a shows graphs for the flow of an aliphatic lubricant, the studies being conducted with capillary and rotation viscosimeters [11], the stress state realized in the latter being close to homogeneous. This graph was redrawn in Fig. 2b in the form of the dependence $\tau^2 = f(\tau\dot{\varepsilon})$. As is evident, the theoretically based Eq. (13) agrees with the experimental data.

A positive feature of the flow equation (13) is that it describes a nonlinear law for the flow of an elastoviscoplastic material using only two rheological constants, each having a real physical sense.

Relations for determining the rate of volume deformation for an elastoviscoplastic medium may be obtained from Eqs. (1) in the form

$$p_c = \frac{2}{3}\,\frac{1+\sigma}{1-2\sigma}\,Ge_v.$$

Differentiating with respect to the time, we obtain

$$\dot{e}_v = \frac{3}{2}\,\frac{1-2\sigma}{1+\sigma}\,\frac{1}{G}\dot{p}_c. \tag{14}$$

From Eq. (14) we see that an elastoviscoplastic medium will be incompressible in any one of the following three cases: for $G = \infty$, for $\sigma = 0.5$, and if $p_c = 0$. It is known that for viscous and for plastic flow $\sigma = 0.5$, therefore the continuity equation $\dot{e}_v = 0$ is satisfied for an elastoviscoplastic medium. In the region where $\sqrt{I_2} < \theta$, an elastoviscoplastic medium is elastic, so that its deformations can be calculated in accord with the equations of elasticity theory. The size and shape of the elastic region are then to be determined from the boundary condition

$$\sqrt{I_2} = \theta = G\sqrt{I_2'}.$$

Thus, depending on the size of the deformations, the entire region of the medium can be partitioned into a region of elastic deformations in which $\sqrt{I_2''} < \theta/G$, and a flow region wherein $\sqrt{I_2''} > \theta/G$. To simplify the calculations in the elastic region we put $\sigma = 0.5$, i.e., we consider the elastoviscoplastic medium to be an elastic-incompressible medium, and this will definitely be so if the deformations during flow of the medium significantly exceed the elastic deformations.

We now obtain the equations for the stationary flow of an elastoviscoplastic medium. For this purpose we use the equation of motion of a continuous medium, assuming the inertia forces to be zero:

$$\rho F + \operatorname{grad} p_c + \operatorname{div} s_{ij} = 0.$$

If in this equation we substitute $p_c = -p$ and the s_{ij} from Eq. (12), we obtain the equation for the stationary flow of an elastoviscoplastic medium:

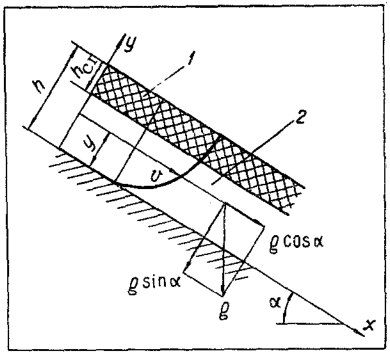

Fig. 3. A model for the flow of an elastoviscoplastic medium along an inclined surface: 1) stopped region; 2) flow region.

$$\rho F - \operatorname{grad} p + \eta_{ef} \operatorname{div} \dot{e}_{ij} + \dot{e}_{ij} \operatorname{div} \eta_{ef} = 0,$$

where $\eta_{ef} = \frac{1}{2}\eta\left(1+\sqrt{1+\frac{4\theta^2}{\eta^2 I_2'}}\right)$. To this equation there is added the equation of continuity.

We pause here to consider the rheological equation of state of a viscoplastic medium. It is readily seen that if in Eq. (7) we substitute the expressions for the stresses from Eqs. (4), we obtain $\lambda = \sqrt{I_2'}$ for a viscoplastic medium, and the rheological equation of state takes the form

$$s_{ij} = 2\left(\frac{\theta}{\sqrt{I_2'}} + \eta\right)\dot{e}_{ij}. \tag{15}$$

For pure shear it becomes the Schwedoff–Bingham equation

$$\tau = \theta + \eta\dot{\varepsilon}.$$

Consequently, for a viscoplastic medium $G \equiv \infty$.

It should be remarked that various points of view exist relative to the question of the behavior of a viscoplastic medium in the region where $\sqrt{I_2'} < \theta$. Thus, for example, Volarovich [12] regards a viscoplastic medium as an "elastic" medium in the region where $\sqrt{I_2'} < \theta$, and Gutkin [13] has even calculated its deformations in this region according to the equations of elasticity theory. As a matter of fact, a viscoplastic medium with $\sqrt{I_2} < \theta$ is a rigid plastic body for which $G = \infty$.

Equation (15) with $\sqrt{I_2'} \ll \theta/\eta$ degenerates into the equation for the flow of a perfectly plastic body, namely,

$$s_{ij} = 2\theta \frac{\dot{e}_{ij}}{\sqrt{I_2'}}.$$

Consequently, a viscoplastic medium, in a region where $\sqrt{I_2} = \theta$, flows like a rigid-plastic medium since in this region the condition $\sqrt{I_2'} = 0$ is not satisfied.

Finally, since for a viscoplastic medium $G = \infty$, it is then a direct result of Eq. (14) that $\dot{e}_V = 0$ for this medium, i.e., it is an incompressible medium. Therefore the equations of a "compressible" viscoplastic medium as written by Kasimov and Mirzadzhanzade [14] and Astrakhan and Grigoryan [15], by analogy with the equations of a compressible viscous liquid, need not be regarded as being equivalent.

Flow in a Circular Pipe. We consider a medium present in a circular pipe. Due to a pressure difference in the medium a tangential stress $\tau_{rz} = \Delta pr/2l$ develops. For $\sigma = 0.5$ we obtain displacements in the form $u_z = u_z(r)$. Hooke's law may then we written

$$\tau_{rz} = \frac{\Delta pr}{2l} = G\frac{du_z}{dr}. \tag{16}$$

If we integrate, taking into account as boundary conditions the "no slip" condition $u_z(R) = 0$ at the pipe wall, we find

$$u_z = \frac{\Delta p}{4Gl}(R^2 - r^2).$$

The radius of the elastic region may be obtained from the boundary condition: for $r = r_0$, $du_z/dr = \theta/G$; we find it to be $r_0 = 2\theta l/\Delta p$. In the region where $r > r_0$ the medium will flow with the speed $v_z = v_z(r)$. If into Eq. (13), written in a cylindrical system of coordinates, we substitute τ_{rz} from Eq. (16), we find, upon integrating and using the "no slip" condition $v_z(R) = 0$, that

$$v_z = \frac{\Delta p}{4\eta l}(R^2 - r^2) + \frac{2\theta^2 l}{\Delta p\eta}\ln\frac{r}{R}.$$

The elastic region moves with the speed

$$v_0 = \frac{\Delta p}{4\eta l}(R^2 - r_0^2) + \frac{2\theta^2 l}{\Delta p\eta}\ln\frac{r_0}{R}.$$

The volumetric outflow of the medium from the tube is given by

$$q = \pi r_0^2 v_0 + 2\pi \int_{r_0}^{R} r v_z dr = \frac{\pi \Delta p R^4}{8\eta l} \left(1 - \frac{r_0^2}{R^2}\right)^2. \tag{17}$$

This equation may be written as

$$\tau_w^2 = \theta^2 + \eta^{1/2}\tau_w^{3/2} D^{1/2}.$$

In Fig. 2b we show the graph of the functional dependence $\tau_w^2 = f(\tau_w^{3/2} D^{1/2})$ for the flow of an aliphatic lubricant in a capillary viscosimeter, drawn in accord with the experimental data [11] given in Fig. 2a. As is evident, the experimental data confirms the theoretical relation (17).

Flow along an Inclined Surface. The flow of a viscoplastic medium under the influence of gravity was studied in [16]. Volarovich and Gutkin, who solved an analogous problem in [17], assumed that in the "elastic" region the pressure was propagated in accord with a hydrostatic law. Finally, these solutions were derived once again in [18]. In all these solutions the thickness of the layer of the medium containing adhesive forces on the inclined surface was equal to

$$h_{cr} = \frac{\theta}{\gamma \sin \alpha}. \tag{18}$$

It is not difficult to see that for $\alpha \to 0$ and $h_{cr} \to \infty$ this contradicts the physics of the phenomenon since an infinitely thick layer cannot be maintained by adhesive forces on an inclined surface because of the action of the very large normal stresses.

We consider equilibrium and flow of a two-dimensional layer of an elastoviscoplastic medium along an inclined surface under the action of the force of gravity (Fig. 3). The equations of equilibrium in a rectangular system of coordinates may be written as follows:

$$p_{xx} = p_{xx}(y), \quad p_{yy} = p_{yy}(y), \quad \tau_{xy} = \tau_{xy}(y),$$

$$\frac{d\tau_{xy}}{dy} + \gamma \sin \alpha = 0, \quad \frac{dp_{yy}}{dy} - \gamma \cos \alpha = 0.$$

Integrating these equations, considering an excess pressure p_0 to be acting on the free surface of the layer, i.e., assuming that $\tau_{xy}(h) = 0$, $p_{yy}(h) = -p_0$, we find

$$\tau_{xy} = \gamma \sin \alpha (h - y), \tag{19}$$

$$p_{yy} = -p_0 - \gamma \cos \alpha (h - y). \tag{20}$$

Since the layer of the medium is infinite, $u_x = u_x(y)$, $u_y = u_y(y)$, $u_z = 0$, and Hooke's law may be written

$$p_{xx} = p_{zz} = \frac{\sigma}{1 - \sigma} p_{yy}, \tag{21}$$

$$\frac{du_y}{dy} = \frac{1 - 2\sigma}{2(1 + \sigma) G} p_{yy}, \tag{22}$$

$$\frac{du_x}{dy} = \frac{1}{G} \tau_{xy}. \tag{23}$$

Substituting Eq. (19) into Eq. (21) and Eq. (20) into Eq. (22) and integrating, and then determining the constants of integration from the "no slip" boundary condition $u_x(0) = 0$, $u_y(0) = 0$ at the surface, we find

$$u_x = \frac{\gamma \sin \alpha}{G} y \left(h - \frac{y}{2}\right),$$

$$u_y = \frac{2\sigma - 1}{2(1 - \sigma) G} y \left[p_0 + \gamma \cos \alpha \left(h - \frac{y}{2}\right)\right].$$

We determine the critical thickness h_{cr} of the layer, which is maintained by the adhesive forces on the inclined surface, from the Mises plasticity condition (7), which after substitution of p_{xx} and p_{zz} from Eq. (21) assumes the form

$$I_2 = \frac{1}{3}\left(\frac{1-2\sigma}{1-\sigma}\right)^2 p_{yy}^2 + \tau_{xy}^2 = \theta^2.$$

Substituting τ_{xy} from Eq. (19) and p_{yy} from Eq. (20), and taking into consideration that $\sqrt{I_2} = \theta$ for $y = h - h_{cr}$, we find

$$h_{cr} = \frac{ap_0 \cos\alpha}{\gamma(a\cos^2\alpha + \sin^2\alpha)}\left[\sqrt{1 + \frac{(\theta^2 - ap_0^2)(a\cos^2\alpha + \sin^2\alpha)}{a^2 p_0^2 \cos^2\alpha}} - 1\right], \quad (24)$$

where $a = \frac{1}{3}\left(\frac{1-2\sigma}{1-\sigma}\right)^2$. For the critical pressure $p_0 = p_{cr}$, $h_{cr} = 0$, we have

$$p_{cr} = \sqrt{3}\,\frac{1-\sigma}{1-2\sigma}\,\theta.$$

Consequently, for $p_0 \geq p_{cr}$, the whole layer will be enveloped by the flow. If the excess pressure $p_0 = 0$, then

$$h_{cr} = \frac{\theta}{\gamma\sqrt{a\cos^2\alpha + \sin^2\alpha}}.$$

For a horizontal surface, $\alpha = 0$, $h_{cr} = h'_{cr}$, for a vertical surface, $\alpha = \pi/2$, $h_{cr} = h''_{cr}$, and we obtain

$$h'_{cr} = \sqrt{3}\,\frac{1-\sigma}{1-2\sigma}\cdot\frac{\theta}{\gamma}, \quad h''_{cr} = \frac{\theta}{\gamma}.$$

For a lubricant at $t = 20°C$, $\sigma = 0.28$ [19], $\theta = 600\ N/m^2$ [5], $h'_{cr} = 18.3$ cm, $h''_{cr} = 6.5$ cm. If there is a layer of the medium on the inclined surface with thickness greater than h_{cr}, then in the region $0 < y < h - h_{cr}$ the medium will flow like a liquid. The components of the flow velocity will be $v_x = v_x(y)$, $v_y = v_z = 0$.

Since the layer is infinite, then upon substituting τ_{xy} from Eq. (19) into Eq. (13) and integrating the resulting equation, using as boundary condition the "no slip" condition of the medium on the inclined surface, i.e., $v_x(0) = 0$, we find the velocity in the form

$$v_x = \frac{\gamma\sin\alpha}{\eta}\, y\left(h - \frac{y}{2}\right) + \frac{\theta^2}{\gamma\eta\sin\alpha}\ln\frac{h-y}{h}.$$

The velocity of the elastic layer on the surface can be obtained from the boundary condition $v_x(h - h_{cr}) = v_0$:

$$v_0 = \frac{\gamma\sin\alpha}{2\eta}(h^2 - h_{cr}^2) + \frac{\theta^2}{\gamma\eta\sin\alpha}\ln\frac{h_{cr}}{h}.$$

The volumetric rate of flow of the medium per unit layer width is equal to

$$q = v_0 h_{cr} + \int_0^{h-h_{cr}} v_x\,dy = \frac{\gamma\sin\alpha(h^3 - h_{cr}^3)}{3\eta} - \frac{\theta^2}{\gamma\eta\sin\alpha}(h - h_{cr}).$$

In the expression h_{cr} is determined from Eq. (24). For $p_0 \geq p_{cr}$, the volumetric outflow rate of the medium is

$$q = \frac{\gamma\sin\alpha h^3}{3\eta} - \frac{\theta^2 h}{\gamma\eta\sin\alpha}.$$

Some cases were treated in [20-21] concerned with the flow of an elastoviscoplastic medium and boundary layer theory.

NOTATION

$D = 4q/\pi R^3$ is the mean velocity gradient, sec^{-1};

$\dot{e}_{ij}$ is the deviator of deformation rates; point stands for partial differentiation with respect to time;

$e_v = \varepsilon_{ii}/3$;
$\dot{e}_v = \dot{\varepsilon}_{ii}/3$;

F	is the mass force;
G	is the shear modulus;
h	is the height of elastoviscoplastic layer;
$I_2 = s_{ij}s_{ij}/2$	is the second invariant of stress deviator;
$I_2' = 2\dot{e}_{ij}\dot{e}_{ij}$	is the second invariant of deformation rate deviator;
l	is the conduit length;
p_{ij}	is the stress tensor;
Δp	is the pressure drop;
q	is the flow rate;
R	is the tube radius;
s_{ij}	is the stress deviator;
u_i	is the shear;
v_i	is the velocity;
x, y, z, r	are the coordinates;
α	is the angle of plane inclination to horizon;
γ, ρ	are the specific weight and density of medium;
δ	is the Kronecker delta;
ε_{ij}	is the deformation tensor;
$\dot{\varepsilon}_{ij}$	is the tensor of deformation rates;
η	is the plastic viscosity;
θ	is the limit shear stress;
σ	is the Poisson coefficient;
τ	is the shear stress;
τ_w	is the shear stress at wall, dyn/cm^2.

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